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THE POSSIBILITY OF STRUCTURALLY STABLE GLOBAL OSCILLATORS OCCURRING WHEN DISSIPATIVE FORCES ARE INTRODUCED INTO DYNAMIC SYSTEMS ASYMPTOTICALLY STABLE IN THE LARGE*

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It is shown that when dissipative interactive forces (of the viscous friction type) are introduced between two thick, mechanical autonomous systems asymptotically stable in the large, the transformation of these systems into a structurally stable global oscillator becomes possible. However, the dimensionality of the configurational space of each such system must be at least two.

We study dynamic systems of the form

$$x^{\cdot\cdot} = F(x, x^{\cdot}),\ x \in \mathbf{R}^n \tag{1}$$

where $\mathbf{R}^n$ is a Euclidean n-dimensional vector space, and a dot denotes differentiation with respect to the independent variable t. The spaces $\mathbf{R}^n = \{x\}, \mathbf{R}^{2n} = \{(x, x^{\cdot})\}$ are called the configurational and phase space of system (1) respectively. The tangent spaces of Euclidean vector spaces are identical with the above spaces. In (1) $F: \mathbf{R}^{2n} \to \mathbf{R}^n = \{x^{\cdot}\}$ represents a smooth mapping. The formulation of system (1) is equivalent to the process of specifying, in its phase space, the vector field $f: \mathbf{R}^{2n} \to \mathbf{R}^{2n}, (x, x^{\cdot}) \mapsto (x^{\cdot}, F(x, x^{\cdot}))$. The dynamic system corresponding to the field f is denoted by $z^{\cdot} = f(z),\ z = (x, x^{\cdot}) \in \mathbf{R}^{2n}$.

Let us introduce some definitions (see e.g. /1, 2/). The structural stability of the system means that the equation $z^{\cdot} = f_0(z)$ has the same structural properties as the equation $z^{\cdot} = f(z)$, provided that the vector field f_0 is derived from f using the C^1-perturbation of f. We shall further call the dynamic system (1) globally asymptotically stable if it has a unique position of equilibrium $x = x_0,\ x^{\cdot} = 0$ and any solution tends, as $t \to \infty$, to this position. Finally, we shall call the dynamic system (1) a global oscillator (GO) if it has a non-trivial periodic solution Γ, and any other solution except the set Σ of zero measure in the phase space tends to Γ as $t \to \infty$. Here we consider only the GO's in which the sets Σ are discs of positive codimensionality, smoothly imbedded in $\mathbf{R}^{2n} = \{(x, x^{\cdot})\}$, and every solution on Σ tends to a unique position of equilibrium $(x_0, 0)$.

It will also be convenient to use mechanical terminology, calling Eqs.(1) the equations of motion of a mechanical system and $F, x^{\cdot}, x^{\cdot\cdot}$ the vectors of forces, velocities and accelerations, respectively.

Let a dynamic system

$$x^{\cdot\cdot} = S(x, x^{\cdot}),\ (x, x^{\cdot}) \in \mathbf{R}^{2n} \tag{2}$$

be structurally stable and globally asymptotically stable.

Connecting two such initially mutually independent identical dynamic systems (2) by means of the dissipative, linearly interacting forces of the "viscous friction" type, we shall consider the following dynamic system:

$$\begin{aligned} x_1^{\cdot\cdot} &= S(x_1, x_1^{\cdot}) + \mu\cdot(x_2^{\cdot} - x_1^{\cdot}),\\ x_2^{\cdot\cdot} &= S(x_2, x_2^{\cdot}) + \mu\cdot(x_1^{\cdot} - x_2^{\cdot}) \end{aligned} \tag{3}$$

where μ is a $(n \times n)$-matrix characterising the dissipative forces. We can assume, without loss of generality, that the matrix is diagonal and non-negative. System (3) is of order $4n$, and is defined in the phase space $\mathbf{R}^{4n} = \{(x_1, x_1^{\cdot}, x_2, x_2^{\cdot})\}$.

The principal aim of this paper is to prove that system (3) can be a structurally stable GO for $n \geqslant 2$. This explains the paradoxical effect which may be of major importance in theoretical mechanics and the mechanics of controlled motion. Thus, let two thick mechanical systems asymptotically stable as a whole be coupled by means of dissipative forces of interaction proportional to the differences between the corresponding velocities. Such an interaction has, by itself, a tendency to equalize these velocities. Nevertheless, the resulting united system may "come to life", i.e. it may convert to a structurally stable GO. If the mechanical system (2) is acted upon only by potential, gyroscopic and definitely dissipative forces, then clearly, the effect in question cannot appear. Thus the effect can only

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materialize when the mechanical system (2) contains either non-conservative or acceleration forces, or forces of a more complex structure.

A similar problem, in which two thick, globally asymptotically stable dynamic systems of general form $z^{\cdot} = f(z)$, $z \in \mathbf{R}^m$ are coupled by a linear, "diffusion" type interaction, was studied in /3, 4/. It was found that the introduction of such an interaction can result in the appearance of a structurally stable GO, provided that the dimension of the phase space is $m \geqslant 3$. In order to use the method of /3/ in the present paper, we had to alter its form considerably. This was necessitated by the fact that the class of autonomous mechanical systems discussed here is narrower than the class of dynamic systems of general form.

Theorem. When $n \geqslant 2$, there exist polynomial mappings $S: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ and non-negative diagonal matrices $\mu = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, such, that

A) the system of Eqs.(2) is structurally stable and globally asymptotically stable,

B) the system of Eqs.(3) represents a structurally stable GO.

Proof. Remembering that it is sufficient to consider the case $n = 2$, we shall prove that there exists a polynomial cubic mapping $S: \mathbf{R}^4 \to \mathbf{R}^2$, $(x, x^{\cdot}) \mapsto S(x, x^{\cdot})$ and a matrix $\mu = \operatorname{diag}(\lambda_1, \lambda_2)$, $\lambda_2 > \lambda_1 \geqslant 0$, such that, firstly, the origin of coordinates $x = x^{\cdot} = 0$ is a thick global attractor of the system of equations $x^{\cdot\cdot} = S(x, x^{\cdot})$ in its phase space $\mathbf{R}^4 = \{(x, x^{\cdot})\}$, and secondly, the vector field

$$(x_1, x_1^{\cdot}, x_2, x_2^{\cdot}) \mapsto (x_1^{\cdot}, S(x_1, x_1^{\cdot}) + \mu\cdot(x_2^{\cdot} - x_1^{\cdot}), \\ x_2^{\cdot}, S(x_2, x_2^{\cdot}) + \mu\cdot(x_1^{\cdot} - x_2^{\cdot})) \tag{4}$$

corresponding to system (3) defines, in the phase space $\mathbf{R}^8 = \{(x_1, x_1^{\cdot}, x_2, x_2^{\cdot})\}$, a structurally stable GO.

The vector field (4) in the phase space $\mathbf{R}^8$ is tangent to the four-dimensional plane $\Delta = \{(x_1, x_1^{\cdot}, x_2, x_2^{\cdot}) \mid x_1 = x_2, x_1^{\cdot} = x_2^{\cdot}\}$, i.e. Δ represents an integral manifold of dynamic system (3) corresponding to the vector field (4), and at Δ itself the vector field has the form

$$(x, x^{\cdot}) \mapsto (x^{\cdot}, S(x, x^{\cdot})), \; x = x_1 = x_2, \; x^{\cdot} = x_1^{\cdot} = x_2^{\cdot} \tag{5}$$

We regard the function S as odd, i.e. $S(-x, -x^{\cdot}) = -S(x, x^{\cdot})$. Then the four-dimensional plane $\Delta^{\perp} = \{(x_1, x_1^{\cdot}, x_2, x_2^{\cdot}) \mid x_1 = -x_2, x_1^{\cdot} = -x_2^{\cdot}\}$ orthogonal to Δ will also represent an integral manifold of dynamic system (3). At the same time, the vector field (4) will transform, on the subspace $\Delta^{\perp}$ itself, into the vector field

$$(x, x^{\cdot}) \mapsto (x^{\cdot}, S(x, x^{\cdot}) - 2\mu\cdot x^{\cdot}), \; x = x_1 = -x_2, \; x^{\cdot} = x_1^{\cdot} = -x_2^{\cdot} \tag{6}$$

This means that the problem has been reduced to that of determining an odd polynomial mapping $S: \mathbf{R}^4 \to \mathbf{R}^2$, $(x, x^{\cdot}) \mapsto S(x, x^{\cdot})$, with the following properties:

1) the origin of coordinates $x = x^{\cdot} = 0$ is a global attractor of the vector field (5) $(x, x^{\cdot}) \mapsto (x^{\cdot}, S(x, x^{\cdot}))$ in the space $\mathbf{R}^4 = \{(x, x^{\cdot})\}$,

2) the vector field (6) $(x, x^{\cdot}) \mapsto (x^{\cdot}, S(x, x^{\cdot}) - 2\mu\cdot x^{\cdot})$ determines the structurally stable GO in $\mathbf{R}^4 = \{(x, x^{\cdot})\}$,

3) the subspace $\Delta^{\perp}$ is a global attractor of the vector field (4) in the space $\mathbf{R}^8 = \{(x_1, x_1^{\cdot}, x_2, x_2^{\cdot})\}$.

Let us carry out, in the space $\mathbf{R}^4 = \{(x, x^{\cdot})\}$ and the corresponding space $\mathbf{R}^2 = \{x^{\cdot}\}$, a linear change of coordinates, the $y = Ax$, $y^{\cdot} = Ax^{\cdot}$, (2×2) -matrix A of which will be specified in more detail later. We shall seek the mapping $S: \mathbf{R}^4 \to \mathbf{R}^2$ in the B $(y, y^{\cdot})$ coordinates in the form of a sum $S = S_1 + S_3$ of the linear mapping S_1: $\mathbf{R}^4 \to \mathbf{R}^2$, $(y, y^{\cdot}) \mapsto S_1(y, y^{\cdot})$ with matrix S_1, and the cubic mapping S_3: $\mathbf{R}^4 \to \mathbf{R}^2$, $S_3(y, y^{\cdot}) = (-(y^{1\cdot})^3, 0)$ where $y = (y^1, y^2)$, $y^{\cdot} = (y^{1\cdot}, y^{2\cdot})$. In the new coordinates $y^{1\cdot}, y^{2\cdot}$ of space $\mathbf{R}^2$ the matrix μ will become the matrix μ_1, similar to μ. Let us put

$$S_1 - 2M_1 = \begin{Vmatrix} -k_1 & 0 & b_{11} & 0 \\ 0 & -k_2 & 0 & -b_{22} \end{Vmatrix} \tag{7}$$

$$M_1 = \begin{Vmatrix} 0 & 0 & \lambda_{11} & \lambda_{12} \\ 0 & 0 & \lambda_{21} & \lambda_{22} \end{Vmatrix}, \quad \mu_1 = \begin{Vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{Vmatrix}$$

where k_1, k_2, b_{11}, b_{22} are positive constants. In $(y, y^{\cdot})$ coordinates the differential equations corresponding to the vector field (6) $(y, y^{\cdot}) \mapsto (y^{\cdot}, S(y, y^{\cdot}) - 2\mu_1\cdot y^{\cdot})$, will become $dy^1/dt = y^{1\cdot}$, $dy^{1\cdot}/dt = -k_1 y^1 + b_{11} y^{1\cdot} - (y^{1\cdot})^3$

$$dy^2/dt = y^{2\cdot}, \; dy^{2\cdot}/dt = -k_2 y^2 - b_{22} y^{2\cdot} \tag{8}$$

The two-dimensional plane $\mathbf{R}^2 = \{(y^1, y^{1\cdot})\}$ in the space $\mathbf{R}^4 = \{(y, y^{\cdot}) = (y^1, y^2, y^{1\cdot}, y^{2\cdot})\}$ will be an attractor of the vector field (6), and in the plane itself the field equations will be equivalent to the Rayleigh equation /5, 6/ $y^{1\cdot\cdot} = -k_1 y^1 + b_{11} y^{1\cdot} - (y^{1\cdot})^3$. Thus if the matrix $S_1 - 2M_1$ has the form (7), then the vector field $(y, y^{\cdot}) \mapsto (y^{\cdot}, S(y, y^{\cdot}) - 2\mu_1\cdot y^{\cdot})$ in the space $\mathbf{R}^4 = \{(y, y^{\cdot})\} \equiv \{(x, x^{\cdot})\}$, equivalent to the field (6) in initial coordinates, will determine the structurally stable GO.

We shall assume that the matrices S_1, μ_1 have the form

$$S_1 = \begin{Vmatrix} -k_1 & 0 & -b_1 & b \\ 0 & -k_2 & -b & -b_2 \end{Vmatrix}, \quad \mu_1 = \frac{1}{2} \begin{Vmatrix} -(b_{11}+b_1) & b \\ -b & (b_{22}-b_2) \end{Vmatrix} \tag{9}$$

where b, b_1, b_2 are certain constants and $b_1 > 0, b_2 > 0$.

We now introduce the function

$$E(y^1, y^2, y^{1\cdot}, y^{2\cdot}) = 1/2\,[(y^{1\cdot})^2 + (y^{2\cdot})^2 + k_1 (y^1)^2 + k_2 (y^2)^2] \tag{10}$$

The following relation will hold on the trajectories of the dynamic system determined by the vector field (5):

$$dE/dt = -b_1 (y^{1\cdot})^2 - b_2 (y^{2\cdot})^2 - (y^{1\cdot})^4 \leqslant 0 \tag{11}$$

Form this we conclude that by virtue of the theorem /7, 8/ on asymptotic stability as a whole, the origin of coordinates $(0, 0)$ is a global attractor of the vector field (5).

Now, provided that the following conditions hold:

$$\begin{gathered} b_{22} > b_{11} + b_1 + b_2 \\ (b_{11} + b_1)(b_{22} - b_2) \leqslant b^2 < (b_{11} + b_{22} + b_1 - b_2)^2/4 \end{gathered} \tag{12}$$

the matrix μ_1 will have different real eigenvalues $\lambda_2 > \lambda_1 \geqslant 0$. Moreover, a basis will exist in the space $\mathbf{R}^2 = \{(y^1, y^2)\}$, of real eigenvectors of the matrix μ_1. We shall take the linear coordinates x^1, x^2 in the space $\mathbf{R}^2 = \{(y^1, y^2)\}$ and $x^1, x^2, x^{1\cdot}, x^{2\cdot}$ in the space $\tilde{\mathbf{R}}^4 = \{(y^1, y^2, y^{1\cdot}, y^{2\cdot})\}$, corresponding to this basis, as the initial coordinates x and $x, x^{\cdot}$. The matrix A is a real matrix of transformation of the coordinates from y^1, y^2 to x^1, x^2.

We shall consider, as an example, the matrices

$$\begin{gathered} S_1 = \begin{Vmatrix} -k_1 & 0 & -\varepsilon & 3 \\ 0 & -k_2 & -3 & -8 \end{Vmatrix}, \quad \mu_1 = \frac{1}{2} \begin{Vmatrix} -(1+\varepsilon) & 3 \\ -3 & 7 \end{Vmatrix} \\ S_1 - 2M_1 = \begin{Vmatrix} -k_1 & 0 & 1 & 0 \\ 0 & -k_2 & 0 & -15 \end{Vmatrix} \end{gathered} \tag{13}$$

If $\varepsilon \in (0, 2/7]$, then all conditions formulated above will hold and when $\varepsilon = 2/7$, one of the eigenvalues $(\lambda_1 = 40/7)$ will be positive and the second eigenvalue will be zero $(\lambda_1 = 0)$.

Let us now check the last condition 3) and show that $\Delta^{\perp}$ is an attracting manifold of the vector field (4). Taking into account the fact that

$$x_1^{\cdot\cdot} + x_2^{\cdot\cdot} = S(x_1, x_1^{\cdot}) + S(x_2, x_2^{\cdot}) \tag{14}$$

we obtain, in $(y, y^{\cdot})$ coordinates, the following expression for the derivative of the function $E(y_1^1 + y_2^1, y_1^2 + y_2^2, y_1^{1\cdot} + y_2^{1\cdot}, y_1^{2\cdot} + y_2^{2\cdot})$ on the trajectories of the vector field (4):

$$dE/dt = -b_1 (y_1^{1\cdot} + y_2^{1\cdot})^2 - b_2 (y_1^{2\cdot} + y_2^{2\cdot})^2 - (y_1^{1\cdot} + y_2^{1\cdot})((y_1^{1\cdot})^3 + (y_2^{1\cdot})^3) \leqslant 0 \tag{15}$$

and the equality sign applies here only when $x_1^{\cdot} = -x_2^{\cdot}$. When condition $x_1^{\cdot} = -x_2^{\cdot}$ holds, the complete phase curves differ from the position of equilibrium, i.e., the origin of coordinates, and can only lie in the plane $\Delta^{\perp}$ of the phase space $\mathbf{R}^8 = \{(x_1, x_1^{\cdot}, x_2, x_2^{\cdot})\}$. Thus, in accordance with the theorem of /7, 8/ mentioned above we conclude that the plane $\Delta^{\perp}$ is a global attractor of the vector field (4).

In conclusion we note that the position of equilibrium and the limit cycle constructed both have a hyperbolic structure.

Notes. 1°. We must remember the possibility that the selfexcited oscillation discussed above may appear in complex, mechanically controlled systems, especially when the controls contain non-conservative (radial correction) forces or accleration forces. It is also important that the effect discussed here is also possible in the case when $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = 0, \lambda_n > 0\ (n \geqslant 2)$, i.e. when the dissipation is introduced through a single channel only.

2°. There is no such model within the manthematical constructions used in the proof of the theorem for the case $n = 1$.

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THE SELFSIMILAR ASYMPTOTIC FORM OF NON-STATIONARY VORTEX FLOWS*

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The hydrodynamic reaction of a viscous incompressible fluid filling a half-space to a rotational impulse applied to its surface is studied. It is established that for a non-stationary flow which occurs in this case, a stable, selfsimilar asymptotic form exists which is independent of the form of the initial perturbation. Asymptotic expressions are obtained for the universal distribution of the meridional velocity near the surface and at infinity.

An analogue of Bernoulli's theorem is established for a class of non-stationary selfsimilar flows of an ideal fluid, and a corresponding integral of motion is obtained for the axisymmetric case.

1. Consider a class of selfsimilar motions of a viscous incompressible fluid whose velocity field is determined by the expression

$$\mathbf{v}=\sqrt{\frac{\gamma}{t}}\,\mathbf{u}\left(\frac{\mathbf{r}}{\sqrt{\gamma t}}\right) \tag{1.1}$$

Here $\mathbf{r}\in R^3$ is the radius vector, t is the time and γ is the characteristic parameter of the problem with the dimensions of circulation. Solutions of this type may describe the asymptotic stage of the reaction of a liquid medium under the action of localized dynamic perturbations.

The system of Navier-Stokes equations for the dimensionless vector function $\mathbf{u}$ will transform, taking (1.1) into account, to the form

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u}-{}^1/_2\mathbf{u}-{}^1/_2\,(\mathbf{a}\cdot\nabla)\,\mathbf{u}=-\nabla p+\varepsilon\Delta\mathbf{u}$$
$$(\nabla\cdot\mathbf{u})=0,\quad \varepsilon=\nu/\gamma,\quad \mathbf{a}=\mathbf{r}/\sqrt{\gamma t},\quad p=tP/(\gamma\sigma)$$

(the operators ∇ and Δ act on $\mathbf{a}$; P is the pressure and σ is the density of the fluid).

2. Let us first consider some general properties of the flows of type (1.1) in the limit when the viscosity becomes vanishingly small $(\nu\ll\gamma)$. In this case the last term of the first equation of (1.2) can be neglected and we can rewrite this equation in a form analogous to Euler's equation in Gromeko-Lamb form

$$\boldsymbol{\omega}\times(\mathbf{u}-{}^1/_2\mathbf{a})=-\nabla\Pi,\quad \Pi=p+{}^1/_2\mathbf{u}^2-{}^1/_2\,(\mathbf{a}\cdot\mathbf{u}) \tag{2.1}$$

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